

# STRONG BOUNDEDNESS, STRONG CONVERGENCE AND GENERALIZED VARIATION

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*To the memory of Naza Tanović-Miller*

**ABSTRACT.** A trigonometric series strongly bounded at two points and with coefficients forming a log-quasidecreasing sequence is necessarily the Fourier series of a function belonging to all  $L^p$  spaces,  $1 \leq p < \infty$ . We obtain new results on strong convergence of Fourier series for functions of generalized bounded variation.

## 1. INTRODUCTION

Extending Hardy-Littlewood's concept of strong  $(C, 1)$  summability to Cesàro methods  $(C, \alpha)$  of order  $\alpha \geq 0$ , Hyslop [8] arrived at his notion of strong convergence. Subsequently, this was successfully applied to the study of trigonometric series in several papers written by N. Tanović-Miller and her co-workers [16, 17, 18, 13, 14]. Strong convergence of trigonometric series attracts attention because of its position between ordinary and absolute convergence [4, 16, 17].

Interesting results about the global behaviour of a series deduced from its behaviour at one or two points were initially related to absolute convergence and obtained by O. Szasz [15] and R. Pippert [10]. The assumption on the coefficients of a series was that their magnitudes form an almost decreasing sequence. The analogues are valid in the case of strong convergence [3].

We introduce new notions of strong boundedness (in Hyslop's sense) and logarithmic quasimonotonicity. We prove that if one requests only strong boundedness of a trigonometric series at two points but imposes logarithmic quasimonotonicity on the magnitudes of its coefficients, then the respective trigonometric series is the Fourier series of a function belonging to all  $L^p$  spaces,  $1 \leq p < \infty$ .

In the area of strong convergence, our attention is turned to Fourier series of regulated functions, i.e., functions belonging to various classes of generalized bounded variation.

## 2. BANACH SPACES OF STRONGLY BOUNDED SEQUENCES

**Definition 2.A.** A sequence of numbers  $\{d_n\}$  is strongly  $(C, 1)$  summable to a limit  $d$  with index  $\lambda > 0$  ( $\lambda$ -strongly  $(C, 1)$  summable to  $d$ ), and we write  $d_n \rightarrow d$   $[C_1]_\lambda$ ,

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if

$$\sum_{k=1}^n |d_k - d|^\lambda = o(n) \text{ as } n \rightarrow \infty.$$

**Definition 2.B.** A sequence of numbers  $\{d_n\}$  is strongly convergent to a limit  $d$  with index  $\lambda > 0$  ( $\lambda$ -strongly convergent to  $d$ ), and we write  $d_n \rightarrow d$   $[I]_\lambda$ , if

- 1)  $d_n \rightarrow d$  as  $n \rightarrow \infty$ ,
- 2)  $\sum_{k=1}^n k^\lambda |d_k - d_{k-1}|^\lambda = o(n)$  as  $n \rightarrow \infty$ , i.e.,  $k(d_k - d_{k-1}) \rightarrow 0$   $[C_1]_\lambda$ .

If  $\lambda = 1$ , we simply denote it by  $[I]$ .

**Definition 2.1.** A sequence of numbers  $\{d_n\}$  is said to be strongly bounded with index  $\lambda > 0$  ( $\lambda$ -strongly bounded), if

- 1)  $d_n = O(1)$  as  $n \rightarrow \infty$ ,
- 2)  $\sum_{k=1}^n k^\lambda |d_k - d_{k-1}|^\lambda = O(n)$  as  $n \rightarrow \infty$ .

If  $\lambda = 1$ , we say that sequence  $\{d_n\}$  is strongly bounded.

The set of  $\lambda$ -strongly bounded sequences is denoted by  $\mathcal{B}^\lambda$ .

It is obvious that every  $\lambda$ -strongly convergent sequence is  $\lambda$ -strongly bounded. The converse does not hold as illustrated by the following examples.

**Example 2.2.** Let a sequence  $\{d_n\}$  be given by  $d_n = \sum_{k=1}^n \frac{(-1)^{k-1}}{k}$  for  $n \in \mathbb{N}$ . This sequence is obviously convergent. Therefore,  $d_n = O(1)$  as  $n \rightarrow \infty$ . For any  $\lambda > 0$ , we have

$$\sum_{k=1}^n k^\lambda |d_k - d_{k-1}|^\lambda = \sum_{k=1}^n k^\lambda \left| \frac{(-1)^{k-1}}{k} \right|^\lambda = \sum_{k=1}^n 1 = n.$$

Hence,  $\{d_n\}$  is  $\lambda$ -strongly bounded but not  $\lambda$ -strongly convergent.

**Example 2.3.** Consider a sequence  $\{d_n\}$  such that

$$d_n = \begin{cases} 1, & \text{if } n = 2^k, k \in \mathbb{N}, \\ 0, & \text{if } n \neq 2^k, k \in \mathbb{N}. \end{cases}$$

This sequence is bounded but it is not convergent since it has two partial limits, 0 and 1. Therefore, it is not  $\lambda$ -strongly convergent. Now, we have  $|d_n - d_{n-1}| = 1$  if  $n = 2^k$  or  $n = 2^k + 1$ . Otherwise,  $d_n - d_{n-1} = 0$ . Let  $0 < \lambda \leq 1$ . Then

$$\sum_{k=1}^n k^\lambda |d_k - d_{k-1}|^\lambda = \sum_{j=0}^{\lfloor \log_2 n \rfloor} [2^{j\lambda} + (2^j + 1)^\lambda] = O(n^\lambda).$$

Therefore, this sequence is  $\lambda$ -strongly bounded for  $0 < \lambda \leq 1$ .

Every  $\mathcal{B}^\lambda$  is a linear space. The next theorem introduces a norm in  $\mathcal{B}^\lambda$  that turns  $\mathcal{B}^\lambda$  into a Banach space.

**Theorem 2.4.**

- i)  $\mathcal{B}^\mu \supseteq \mathcal{B}^\lambda$  for  $0 < \mu < \lambda$ .
- ii) For  $d = \{d_n\}_{n=1}^\infty \in \mathcal{B}^\lambda$ ,  $\lambda \geq 1$ , let

$$\|d\|_{\mathcal{B}^\lambda} = \sup_n |d_n| + \sup_n \left( \frac{1}{n} \sum_{k=1}^n k^\lambda |d_k - d_{k-1}|^\lambda \right)^{\frac{1}{\lambda}}.$$

$\|\cdot\|_{\mathcal{B}^\lambda}$  is a norm on  $\mathcal{B}^\lambda$ ,  $\lambda \geq 1$ .

iii)  $\mathcal{B}^\lambda$ ,  $\lambda \geq 1$ , is a Banach space under the norm given in ii).

*Proof.*

i) This is an immediate consequence of Hölder's inequality

$$\sum_{k=1}^n k^\mu |d_k - d_{k-1}|^\mu \leq \left( \sum_{k=1}^n k^\lambda |d_k - d_{k-1}|^\lambda \right)^{\frac{\mu}{\lambda}} \left( \sum_{k=1}^n 1 \right)^{1-\frac{\mu}{\lambda}} = O(n)$$

for  $0 < \mu < \lambda$ .

ii) It is obvious that  $\|d\|_{\mathcal{B}^\lambda} \geq 0$  and that the equality holds if and only if  $d = \{0\}_{n=1}^\infty$ . If  $\alpha$  is an arbitrary complex number and  $\alpha d := \{\alpha d_n\}_{n=1}^\infty$ , then

$$\begin{aligned} \|\alpha d\|_{\mathcal{B}^\lambda} &= \sup_n |\alpha d_n| + \sup_n \left( \frac{1}{n} \sum_{k=1}^n k^\lambda |\alpha d_k - \alpha d_{k-1}|^\lambda \right)^{\frac{1}{\lambda}} \\ &= |\alpha| \left( \sup_n |d_n| + \sup_n \left( \frac{1}{n} \sum_{k=1}^n k^\lambda |d_k - d_{k-1}|^\lambda \right)^{\frac{1}{\lambda}} \right) = |\alpha| \|d\|_{\mathcal{B}^\lambda}. \end{aligned}$$

If  $d^{(1)} = \{d_n^{(1)}\}_{n=1}^\infty$  and  $d^{(2)} = \{d_n^{(2)}\}_{n=1}^\infty$  are two  $\lambda$ -strongly bounded sequences,  $\lambda \geq 1$ , and  $d^{(1)} + d^{(2)} := \{d_n^{(1)} + d_n^{(2)}\}_{n=1}^\infty$ , then by Minkowski's inequality we get

$$\begin{aligned} & \left| d_n^{(1)} + d_n^{(2)} \right| + \left( \frac{1}{n} \sum_{k=1}^n k^\lambda \left| (d_k^{(1)} + d_k^{(2)}) - (d_{k-1}^{(1)} + d_{k-1}^{(2)}) \right|^\lambda \right)^{\frac{1}{\lambda}} \\ &= \left| d_n^{(1)} + d_n^{(2)} \right| + \left( \frac{1}{n} \sum_{k=1}^n k^\lambda \left| (d_k^{(1)} - d_{k-1}^{(1)}) + (d_k^{(2)} - d_{k-1}^{(2)}) \right|^\lambda \right)^{\frac{1}{\lambda}} \\ &\leq \left| d_n^{(1)} \right| + \left| d_n^{(2)} \right| + \left( \frac{1}{n} \sum_{k=1}^n k^\lambda \left| d_k^{(1)} - d_{k-1}^{(1)} \right|^\lambda \right)^{\frac{1}{\lambda}} + \left( \frac{1}{n} \sum_{k=1}^n k^\lambda \left| d_k^{(2)} - d_{k-1}^{(2)} \right|^\lambda \right)^{\frac{1}{\lambda}}. \end{aligned}$$

Hence,

$$\begin{aligned} & \left\| d^{(1)} + d^{(2)} \right\|_{\mathcal{B}^\lambda} \\ &\leq \sup_n \left| d_n^{(1)} \right| + \sup_n \left( \frac{1}{n} \sum_{k=1}^n k^\lambda \left| d_k^{(1)} - d_{k-1}^{(1)} \right|^\lambda \right)^{\frac{1}{\lambda}} + \sup_n \left| d_n^{(2)} \right| + \sup_n \left( \frac{1}{n} \sum_{k=1}^n k^\lambda \left| d_k^{(2)} - d_{k-1}^{(2)} \right|^\lambda \right)^{\frac{1}{\lambda}} \\ &= \left\| d^{(1)} \right\|_{\mathcal{B}^\lambda} + \left\| d^{(2)} \right\|_{\mathcal{B}^\lambda}. \end{aligned}$$

Thus,  $\mathcal{B}^\lambda$ ,  $\lambda \geq 1$ , is a normed linear space.

iii) It remains to check that  $\mathcal{B}^\lambda$ ,  $\lambda \geq 1$ , is complete. Let  $d^{(1)}, d^{(2)}, \dots$  be a Cauchy sequence in  $\mathcal{B}^\lambda$ . Now,

$$(\forall \varepsilon > 0) (\exists n_0 \in \mathbb{N}) (\forall m > n \geq n_0) \left\| d^{(m)} - d^{(n)} \right\|_{\mathcal{B}^\lambda} < \frac{\varepsilon}{3}.$$

Note that

$$\left\| d^{(m)} - d^{(n)} \right\|_{l^\infty} \leq \left\| d^{(m)} - d^{(n)} \right\|_{\mathcal{B}^\lambda}.$$

Thus,  $d^{(n)}$  is a Cauchy sequence in  $l^\infty$ . Since  $l^\infty$  is a Banach space, there exists  $d = \{d_i\}_{i=1}^\infty \in l^\infty$  such that  $\|d^{(n)} - d\|_{l^\infty} \rightarrow 0$  ( $n \rightarrow \infty$ ). Hence, for  $\varepsilon > 0$  chosen above,

$$(2.1) \quad (\exists n_0^* \in \mathbb{N}) (\forall n \geq n_0^*) \|d^{(n)} - d\|_{l^\infty} < \frac{\varepsilon}{3}.$$

Moreover,

$$(2.2) \quad (\forall k \in \mathbb{N}) (\exists n_k \in \mathbb{N}) (\forall n \geq n_k) \left| d_k^{(n)} - d_k \right| < \frac{\varepsilon}{6(k+1)2^{\frac{k+1}{\lambda}}}.$$

Let us show that  $\{d^{(n)}\}_{n=1}^\infty$  converges to  $d = \{d_i\}_{i=1}^\infty$  in  $\mathcal{B}^\lambda$ . Take an arbitrary  $i \in \mathbb{N}$  and fix it. Put  $n_i^* = \max\{n_0^*, n_1, n_2, \dots, n_i\} \in \mathbb{N}$ . To simplify notation, let us put  $\sigma(i, d) = \left(\sum_{k=1}^i k^\lambda |d_k - d_{k-1}|^\lambda\right)^{\frac{1}{\lambda}}$ . Minkowski's inequality and (2.2) yield

$$\begin{aligned} \sigma(i, d^{(n_i^*)} - d) &= \left(\frac{1}{i} \sum_{k=1}^i k^\lambda \left| (d_k^{(n_i^*)} - d_k) + (d_{k-1} - d_{k-1}^{(n_i^*)}) \right|^\lambda\right)^{\frac{1}{\lambda}} \\ &\leq \left(\frac{1}{i} \sum_{k=1}^i k^\lambda |d_k^{(n_i^*)} - d_k|^\lambda\right)^{\frac{1}{\lambda}} + \left(\frac{1}{i} \sum_{k=1}^i k^\lambda |d_{k-1} - d_{k-1}^{(n_i^*)}|^\lambda\right)^{\frac{1}{\lambda}} \\ &\leq \left(\frac{1}{i} \sum_{k=1}^i k^\lambda \frac{\varepsilon^\lambda}{6^\lambda (k+1)^\lambda 2^{k+1}}\right)^{\frac{1}{\lambda}} + \left(\frac{1}{i} \sum_{k=1}^i k^\lambda \frac{\varepsilon^\lambda}{6^\lambda k^\lambda 2^k}\right)^{\frac{1}{\lambda}} \leq \frac{\varepsilon}{3}. \end{aligned}$$

Taking into account (2.1), we get

$$\begin{aligned} \sigma(i, d^{(m)} - d) &\leq \sigma(i, d^{(m)} - d^{(n_i^*)}) + \sigma(i, d^{(n_i^*)} - d) \\ &\leq \|d^{(m)} - d^{(n_i^*)}\|_{\mathcal{B}^\lambda} + \frac{\varepsilon}{3} < \frac{2\varepsilon}{3} \quad (\forall m \geq n_0^{**} = \max\{n_0, n_0^*\} \text{ and } \forall i \in \mathbb{N}) \end{aligned}$$

Therefore,

$$\|d^{(m)} - d\|_{\mathcal{B}^\lambda} = \|d^{(m)} - d\|_{l^\infty} + \sup_i \sigma(i, d^{(m)} - d) < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon \quad (\forall m \geq n_0^{**}).$$

Finally,

$$\|d\|_{\mathcal{B}^\lambda} \leq \|d - d^{(n_0^{**})}\|_{\mathcal{B}^\lambda} + \|d^{(n_0^{**})}\|_{\mathcal{B}^\lambda} < \infty.$$

Hence,  $d \in \mathcal{B}^\lambda$ . □

### 3. LOCAL TO GLOBAL: BEHAVIOUR OF TRIGONOMETRIC SERIES OF A SPECIAL TYPE

**Definition 3.A.** A sequence of positive numbers  $\{d_n\}$  is said to be almost decreasing if there exists a constant  $M$  such that  $d_{n+1} \leq M d_n$  holds for every  $n \in \mathbb{N}$ .  $M$  is the index of almost monotonicity of  $\{d_n\}$ . The space of almost decreasing sequences with index  $M$  is denoted by  $\mathcal{A}_M \mathcal{M}$ . If  $d_{n+1} \leq M d_n$  holds true starting from some integer  $n > 1$ , the corresponding space is denoted by  $\mathcal{G} \mathcal{A}_M \mathcal{M}$ .

*Remark 3.1.* Note that  $\mathcal{A}_1 \mathcal{M} = \mathcal{M}$  is the space of decreasing sequences.

The role of almost decreasing sequences is nicely illustrated by the following theorem.

**Theorem 3.B.** Let  $\rho_n = \sqrt{a_n^2 + b_n^2}$ ,  $n \in \mathbb{N}$ , form an almost decreasing sequence and let

$$\sum A_n(x) \equiv \frac{a_0}{2} + \sum a_n \cos nx + b_n \sin nx,$$

$$\sum B_n(x) \equiv \sum a_n \sin nx - b_n \cos nx.$$

(a) (cf. [10, Theorem 2]) If one of the series  $\sum A_n(x)$  or  $\sum B_n(x)$  is absolutely convergent at two points  $x_0$  and  $x_1$  with  $|x_0 - x_1| \not\equiv 0 \pmod{\pi}$ , then  $\sum \rho_n < \infty$ .

(b) (cf. [3, Theorem 2.2]) If one of the series  $\sum A_n(x)$  or  $\sum B_n(x)$  is  $[I]_\lambda$ ,  $\lambda \geq 1$ , convergent at two points  $x_0$  and  $x_1$  with  $|x_0 - x_1| \not\equiv 0 \pmod{\pi}$ , then  $n\rho_n \rightarrow 0$   $[C_1]_\lambda$ . If  $\lambda > 1$ , then  $\sum A_n(x)$  is the Fourier series of a function  $f \in \bigcap_{1 \leq p < \infty} L^p$ ,  $[I]_\lambda$  convergent to  $f$  a.e., and  $\sum B_n(x)$  is the Fourier series of its conjugate function  $\tilde{f}$ ,  $[I]_\lambda$  convergent to  $\tilde{f}$  a.e.

In the next theorem, we shall replace the condition of  $\lambda$ -strong convergence by  $\lambda$ -strong boundedness. A series is said to be  $\lambda$ -strongly bounded if the sequence of its partial sums is  $\lambda$ -strongly bounded.

**Theorem 3.2.** Let  $\rho_n = \sqrt{a_n^2 + b_n^2}$ ,  $n \in \mathbb{N}$ , form an almost decreasing sequence. If one of the series

$$\sum A_n(x) \equiv \frac{a_0}{2} + \sum a_n \cos nx + b_n \sin nx,$$

$$\sum B_n(x) \equiv \sum a_n \sin nx - b_n \cos nx$$

is  $\lambda$ -strongly bounded,  $\lambda > 1$ , at two points  $x_0$  and  $x_1$ ,  $|x_0 - x_1| \not\equiv 0 \pmod{\pi}$ , then  $\sum A_n(x)$  and  $\sum B_n(x)$  are Fourier series of functions  $f$ ,  $\tilde{f}$ , resp., belonging to  $L^p$  for each  $1 \leq p < \infty$ .

*Proof.* Let  $\theta_n$  be chosen such that  $\sin \theta_n = \frac{a_n}{\rho_n}$  and  $\cos \theta_n = \frac{b_n}{\rho_n}$ . Then,  $\sum A_n(x)$  may be written in the form  $\sum \rho_n \sin(nx + \theta_n)$  and  $\sum B_n(x) = -\sum \rho_n \cos(nx + \theta_n)$ . Assume that  $\sum A_n(x)$  is  $\lambda$ -strongly bounded at two points:

$$(3.1) \quad \sum_{k=1}^n k^\lambda \rho_k^\lambda |\sin(kx_i + \theta_k)|^\lambda = O(n) \text{ as } n \rightarrow \infty, \text{ for } i = 0, 1.$$

Let  $h = x_0 - x_1$ . Then  $nh = (nx_0 + \theta_n) - (nx_1 + \theta_n)$  and

$$\sin nh = \sin(nx_0 + \theta_n) \cos(nx_1 + \theta_n) - \cos(nx_0 + \theta_n) \sin(nx_1 + \theta_n).$$

Therefore,

$$|\sin nh|^\lambda \leq 2^\lambda \left( |\sin(nx_0 + \theta_n)|^\lambda + |\sin(nx_1 + \theta_n)|^\lambda \right).$$

The last inequality and (3.1) imply

$$(3.2) \quad \sum_{k=1}^n k^\lambda \rho_k^\lambda |\sin kh|^\lambda = O(n) \text{ as } n \rightarrow \infty.$$

Let  $\{\rho_k\} \in \mathcal{GA}_M \mathcal{M}$ ,  $M > 1$ . There exists  $K \in \mathbb{N}$  such that

$$\rho_{k-1} \geq \frac{1}{M} \rho_k \text{ for } k \geq K.$$

One has

$$\begin{aligned}
(k-1)\rho_{k-1}|\sin(k-1)h| + k\rho_k|\sin kh| &\geq \frac{1}{M}(k-1)\rho_k|\sin(k-1)h| + k\rho_k|\sin kh| \\
&\geq \frac{k-1}{Mk}k\rho_k(|\sin(k-1)h| + |\sin kh|) \\
&\geq \frac{1-\epsilon}{M}k\rho_k(|\sin(k-1)h| + |\sin kh|)
\end{aligned}$$

for  $k > k_0 = \max\{K, \lfloor \frac{1}{\epsilon} \rfloor\}$ ,  $\epsilon > 0$  arbitrarily small. This and

$$\begin{aligned}
|\sin(k-1)h| + |\sin kh| &\geq \sin^2(k-1)h + \sin^2 kh \\
&= 1 - \cos h \cos(2k-1)h \geq 1 - |\cos h|
\end{aligned}$$

yield

$$(k-1)\rho_{k-1}|\sin(k-1)h| + k\rho_k|\sin kh| \geq M_1 k \rho_k$$

for  $k > k_0$ , where  $M_1 = M_1(h) > 0$ . Hence,

$$\begin{aligned}
\sum_{k=1}^n k^\lambda \rho_k^\lambda &= \sum_{k=1}^{k_0} k^\lambda \rho_k^\lambda + \sum_{k=k_0+1}^n k^\lambda \rho_k^\lambda \\
&\leq \sum_{k=1}^{k_0} k^\lambda \rho_k^\lambda + \frac{2^\lambda}{M_1^\lambda} \sum_{k=k_0+1}^n \left[ (k-1)^\lambda \rho_{k-1}^\lambda |\sin(k-1)h|^\lambda + k^\lambda \rho_k^\lambda |\sin kh|^\lambda \right].
\end{aligned}$$

Since the first summand in the last line is a finite sum and the second one is  $O(n)$  as  $n \rightarrow \infty$  by (3.2), we get

$$u_{n,\lambda} := \sum_{k=1}^n k^\lambda \rho_k^\lambda = O(n) \text{ as } n \rightarrow \infty.$$

Now, let  $p > \max\left\{\frac{\lambda}{\lambda-1}, 2\right\}$  and  $\frac{1}{q} = 1 - \frac{1}{p}$ . It is straightforward that  $1 < q < \min\{\lambda, 2\}$ . Notice that  $\sum_{k=1}^n k^\lambda \rho_k^\lambda = O(n)$  implies  $u_{n,q} = \sum_{k=1}^n k^q \rho_k^q = O(n)$ .

Abel's partial summation formula gives us

$$\begin{aligned}
(3.3) \quad \sum_{k=1}^n \rho_k^q &= \sum_{k=1}^n \frac{k^q \rho_k^q}{k^q} = \sum_{k=1}^n \frac{u_{k,q} - u_{k-1,q}}{k^q} = \frac{u_{n,q}}{n^q} + \sum_{k=1}^{n-1} u_{k,q} \left( \frac{1}{k^q} - \frac{1}{(k+1)^q} \right) \\
&= O\left(\frac{1}{n^{q-1}}\right) + O\left(\sum_{k=1}^{n-1} \frac{1}{k^q}\right) = O(1) \text{ as } n \rightarrow \infty.
\end{aligned}$$

By the Hausdorff-Young theorem [24, (2.3), (ii), p. 101], there exists  $f \in L^p$  such that  $\sum A_n(x)$  is the Fourier series of  $f$ . This and the uniqueness property of Fourier series yield that  $f$  belongs to all  $L^p$  spaces,  $1 \leq p < \infty$ . Then  $\sum B_n(x)$  is the Fourier series of  $\tilde{f} \in \bigcap_{1 \leq p < \infty} L^p$ .

The proof is analogous if  $\sum B_n(x)$  is  $\lambda$ -strongly bounded at  $x_0, x_1$  or if  $\sum A_n(x)$  is  $\lambda$ -strongly bounded at  $x_0$  and  $\sum B_n(x)$  at  $x_1$ .  $\square$

**Definition 3.C.** A sequence of positive numbers  $\{d_n\}$  is said to be quasi decreasing if there exists  $\alpha > 0$  such that  $\{d_n/n^\alpha\}$  is a decreasing sequence starting from some integer  $n \geq 1$ .  $\alpha$  is the index of quasimonotonicity of  $\{d_n\}$ . The space of quasi decreasing sequences with index  $\alpha$  is denoted by  $\mathcal{Q}_\alpha \mathcal{M}$ .

As an application of the concept introduced by Definition 3.C, we cite the next result.

**Theorem 3.D.** ([3, Theorem 3.1]) *Let  $\rho_n = \sqrt{a_n^2 + b_n^2}$ ,  $n \in \mathbb{N}$ , form a quasi decreasing sequence with index  $0 < \alpha < 1$ . Let a trigonometric series  $\frac{a_0}{2} + \sum a_n \cos nx + b_n \sin nx$  be strongly convergent at two points  $x_0$  and  $x_1$  with  $|x_0 - x_1| \not\equiv 0 \pmod{\pi}$ . Then this series and its conjugate are Fourier series, strongly convergent a.e.*

Having in mind that the classes of  $\lambda$ -strongly bounded sequences,  $\lambda > 1$ , are contained in the class of strongly bounded (i.e., 1-strongly bounded) sequences, we turn a closer attention to the latter case.

We shall consider a new class of logarithmic quasi decreasing sequences.

**Definition 3.3.** A sequence of positive numbers  $\{d_n\}$  is said to be logarithmic quasi decreasing if there exists  $\beta > 0$  such that  $\{d_n / \log^\beta n\}$  is a decreasing sequence starting from some integer  $n \geq 2$ .  $\beta$  is the index of logarithmic quasimonotonicity of  $\{d_n\}$ . The set of logarithmic quasi decreasing sequences with index  $\beta$  is denoted by  $\mathcal{L}_\beta \mathcal{QM}$ .

**Theorem 3.4.** *Let  $\rho_n = \sqrt{a_n^2 + b_n^2}$ ,  $n \in \mathbb{N}$ , form a logarithmic quasi decreasing sequence with index  $\beta > 1$  ( $\rho_n \in \mathcal{L}_\beta \mathcal{QM}$ ). If one of the series*

$$\begin{aligned} \sum A_n(x) &\equiv \frac{a_0}{2} + \sum a_n \cos nx + b_n \sin nx, \\ \sum B_n(x) &\equiv \sum a_n \sin nx - b_n \cos nx \end{aligned}$$

*is strongly bounded at two points  $x_0, x_1$ ,  $|x_0 - x_1| \not\equiv 0 \pmod{\pi}$ , then  $\sum \frac{n\rho_n^2}{\log^\beta n} < \infty$ .  $\sum A_n(x)$  and  $\sum B_n(x)$  are Fourier series of functions  $f, \tilde{f}$  which belong to  $L^p$  for each  $1 \leq p < \infty$ .*

*Proof.* Since  $\{\rho_k\} \in \mathcal{L}_\beta \mathcal{QM}$ , we have

$$\rho_{k-1} \geq \frac{\log^\beta(k-1)}{\log^\beta k} \rho_k \text{ for } k \geq K \geq 2.$$

Reasoning as in the proof of Theorem 3.2, we get

$$u_n := \sum_{k=1}^n k \rho_k = O(n) \text{ as } n \rightarrow \infty.$$

Now, for any  $\alpha > 1$ , one has

$$\begin{aligned} (3.4) \quad \sum_{k=2}^n \frac{\rho_k}{\log^\alpha k} &= \sum_{k=2}^n \frac{k \rho_k}{k \log^\alpha k} = \sum_{k=2}^n \frac{u_k - u_{k-1}}{k \log^\alpha k} \\ &= \frac{u_n}{n \log^\alpha n} - \frac{u_1}{2 \log^\alpha 2} + \sum_{k=2}^{n-1} u_k \left( \frac{1}{k \log^\alpha k} - \frac{1}{(k+1) \log^\alpha (k+1)} \right). \end{aligned}$$

Obviously

$$(3.5) \quad \frac{u_n}{n \log^\alpha n} = o(1) \text{ as } n \rightarrow \infty.$$

Notice that

$$\frac{1}{k \log^\alpha k} - \frac{1}{(k+1) \log^\alpha(k+1)} = \frac{1}{\xi_k^2 \log^\alpha \xi_k} \left(1 + \frac{\alpha}{\log \xi_k}\right),$$

where  $\xi_k \in (k, k+1)$ . From  $\frac{1}{\xi_k^2 \log^\alpha \xi_k} \left(1 + \frac{\alpha}{\log \xi_k}\right) < \frac{1}{k^2 \log^\alpha k} \left(1 + \frac{\alpha}{\log 2}\right)$  and  $u_k = O(k)$ , we get

$$u_k \left( \frac{1}{k \log^\alpha k} - \frac{1}{(k+1) \log^\alpha(k+1)} \right) = O \left( \frac{1}{k \log^\alpha k} \right).$$

Thus,

$$(3.6) \quad \sum_{k=2}^{n-1} u_k \left( \frac{1}{k \log^\alpha k} - \frac{1}{(k+1) \log^\alpha(k+1)} \right) = O \left( \sum_{k=2}^{n-1} \frac{1}{k \log^\alpha k} \right) = O(1)$$

as  $n \rightarrow \infty$ .

The relations (3.4), (3.5) and (3.6) yield

$$(3.7) \quad \sum_{k=2}^n \frac{\rho_k}{\log^\alpha k} = O(1) \text{ as } n \rightarrow \infty, \text{ for } \alpha > 1.$$

In particular, the series  $\sum_{k=2}^\infty \frac{\rho_k}{\log^\beta k}$  is convergent. This and the fact that the sequence  $\left\{ \frac{\rho_k}{\log^\beta k} \right\}$  is decreasing yield  $\frac{k\rho_k}{\log^\beta k} = o(1)$  as  $k \rightarrow \infty$  by Olivier's theorem. Now,

$$\begin{aligned} \sum_{k=2}^n \frac{k\rho_k^2}{\log^\beta k} &= \sum_{k=2}^n \frac{\rho_k}{\log^\beta k} (u_k - u_{k-1}) \\ &= \frac{u_n \rho_n}{\log^\beta n} + \sum_{k=2}^{n-1} u_k \left( \frac{\rho_k}{\log^\beta k} - \frac{\rho_{k+1}}{\log^\beta(k+1)} \right) - \frac{\rho_2 u_1}{\log^\beta 2} \\ &\leq o(1) + C \sum_{k=2}^{n-1} k \left( \frac{\rho_k}{\log^\beta k} - \frac{\rho_{k+1}}{\log^\beta(k+1)} \right) \\ &= o(1) + C \sum_{k=2}^{n-1} \left( \frac{k\rho_k}{\log^\beta k} - \frac{(k+1)\rho_{k+1}}{\log^\beta(k+1)} \right) + C \sum_{k=2}^{n-1} \frac{\rho_{k+1}}{\log^\beta(k+1)} \\ &= o(1) + C \left( \frac{2\rho_2}{\log^\beta 2} - \frac{n\rho_n}{\log^\beta n} \right) + C \sum_{k=2}^{n-1} \frac{\rho_{k+1}}{\log^\beta(k+1)} = O(1) \text{ as } n \rightarrow \infty. \end{aligned}$$

This proves the first assertion

$$(3.8) \quad \sum_{k=2}^\infty \frac{k\rho_k^2}{\log^\beta k} < \infty.$$

Concerning the second assertion, (3.8) and the Riesz-Fischer theorem yield that  $\sum A_n(x)$  and  $\sum B_n(x)$  are Fourier series of  $f, \tilde{f} \in L^2$ .



Now, let  $p > 2$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . Obviously,  $1 < q < 2$ . As above,

$$(3.9) \quad \begin{aligned} \sum_{k=1}^n \rho_k^q &= \sum_{k=1}^n \frac{1}{k^q} k^q \rho_k^q = \sum_{k=1}^{n-1} \left( \Delta \frac{1}{k^q} \right) \sum_{i=1}^k i^q \rho_i^q + \frac{1}{n^q} \sum_{i=1}^n i^q \rho_i^q \\ &= O \left( \sum_{k=1}^{n-1} \frac{1}{k^{q+1}} \sum_{i=1}^k i^q \rho_i^q \right) + \frac{1}{n^q} \sum_{i=1}^n i^q \rho_i^q. \end{aligned}$$

Since  $\frac{k \rho_k}{\log^\beta k} = o(1)$  as  $k \rightarrow \infty$ , we have that  $k^q \rho_k^q = o(\log^{\beta q} k)$ . Therefore,

$$\sum_{i=1}^m i^q \rho_i^q = O(m \log^{\beta q} m) \text{ for } m \in \mathbb{N}.$$

The last equality, relation (3.9) and the fact that  $q > 1$  yield

$$\sum_{k=1}^n \rho_k^q = O \left( \sum_{k=1}^{n-1} \frac{\log^{\beta q} k}{k^q} \right) + O \left( \frac{\log^{\beta q} n}{n^{q-1}} \right) = O(1) \text{ as } n \rightarrow \infty.$$

Thus,  $f, \tilde{f} \in \bigcap_{1 \leq p < \infty} L^p$  (cf. the end of the proof of Theorem 3.2).  $\square$

*Remark 3.5.* Pointwise convergence a.e. of the series  $\sum A_n(x)$  and  $\sum B_n(x)$  in Theorem 3.4 follows, of course, from the Carleson-Hunt theorem. However, the Kolmogorov-Selyverstov-Plessner theorem [6, p. 332] already serves the purpose since

$$\sum_{k=2}^{\infty} \rho_k^2 \log k < \sum_{k=2}^{\infty} \frac{k \rho_k^2}{\log^\beta k} < \infty$$

by (3.8).

The following remark concerns the relationship between various sequence spaces considered in this paper.

*Remark 3.6.* For  $0 < M_1 < 1 < M_2$ , one has

$$\mathcal{A}_{M_1} \mathcal{M} \subset \mathcal{M} \subset \bigcap_{\beta > 0} \mathcal{L}_\beta \mathcal{Q} \mathcal{M} \subset \bigcup_{\beta > 0} \mathcal{L}_\beta \mathcal{Q} \mathcal{M} \subset \bigcap_{\alpha > 0} \mathcal{Q}_\alpha \mathcal{M} \subset \bigcup_{\alpha > 0} \mathcal{Q}_\alpha \mathcal{M} \subset \mathcal{G} \mathcal{A}_{M_2} \mathcal{M}.$$

*Proof.* It is obvious that  $\mathcal{A}_{M_1} \mathcal{M} \subset \mathcal{M} \subset \bigcap_{\beta > 0} \mathcal{L}_\beta \mathcal{Q} \mathcal{M}$  since  $0 < M_1 < 1 < \frac{\log^\beta(n+1)}{\log^\beta n}$  for any  $\beta > 0$  and  $n \in \mathbb{N}$ . The inclusions  $\bigcap_{\beta > 0} \mathcal{L}_\beta \mathcal{Q} \mathcal{M} \subset \bigcup_{\beta > 0} \mathcal{L}_\beta \mathcal{Q} \mathcal{M}$  and  $\bigcap_{\alpha > 0} \mathcal{Q}_\alpha \mathcal{M} \subset \bigcup_{\alpha > 0} \mathcal{Q}_\alpha \mathcal{M}$  are trivial. The inclusion  $\bigcup_{\alpha > 0} \mathcal{Q}_\alpha \mathcal{M} \subset \mathcal{G} \mathcal{A}_{M_2} \mathcal{M}$  follows from  $\frac{(n+1)^\alpha}{n^\alpha} < M_2$  for any  $\alpha > 0$ ,  $M_2 > 1$  and sufficiently large  $n \in \mathbb{N}$ . Finally, to establish  $\bigcup_{\beta > 0} \mathcal{L}_\beta \mathcal{Q} \mathcal{M} \subset \bigcap_{\alpha > 0} \mathcal{Q}_\alpha \mathcal{M}$ , it is enough to check that

$$(3.10) \quad \frac{\log^\beta(n+1)}{\log^\beta n} \leq \frac{(n+1)^\alpha}{n^\alpha}$$

holds true for  $\beta > 0$ ,  $\alpha > 0$  and  $n$  sufficiently large. The last inequality is equivalent to

$$\frac{\log(n+1)}{\log n} \leq \left(1 + \frac{1}{n}\right)^\gamma$$

where we put  $\gamma = \frac{\alpha}{\beta} > 0$ . Subtracting 1 from both sides, we get

$$\frac{\log\left(1 + \frac{1}{n}\right)}{\log n} \leq \left(1 + \frac{1}{n}\right)^\gamma - 1.$$

According to Taylor's formula, the left hand side is equal to  $\frac{1}{n \log n} - \frac{1}{2n^2 \log n} + O\left(\frac{1}{n^3 \log n}\right)$ , while the right hand side is equal to  $\frac{\gamma}{n} + \frac{\gamma(\gamma-1)}{2n^2} + O\left(\frac{1}{n^3}\right)$ . Therefore, inequality (3.10) holds true for  $\beta > 0$ ,  $\alpha > 0$  and sufficiently large  $n$ .  $\square$

*Remark 3.7.* In Remark 3.6 we are actually dealing with equivalence classes. Namely, while proving the inclusions, we suppose that  $\{d_k\}_{k \geq k_0}$  and  $\{d_k\}_{k \geq k_1}$ ,  $k_0 \neq k_1$ , represent the same sequence.

*Remark 3.8.* We have seen in Theorem 3.2 that if  $\{\rho_k\} \in \mathcal{GA}_M \mathcal{M}$ ,  $M > 1$ , then a mere  $\lambda$ -boundedness,  $\lambda > 1$ , of the series  $\sum A_n(x)$  or  $\sum B_n(x)$  at two distinct points is sufficient to conclude that these series are Fourier series of functions  $f, \tilde{f}$  belonging to all  $L^p$  spaces,  $1 \leq p < \infty$ . In the case  $\lambda = 1$ , the same conclusion is valid under a stronger assumption  $\rho_k \in \mathcal{L}_\beta \mathcal{QM}$ ,  $\beta > 1$ . For intermediate classes  $\mathcal{Q}_\alpha \mathcal{M}$ ,  $\alpha > 0$ , the same techniques of the proof yield the following theorem.

**Theorem 3.9.** *Let  $\{\rho_k\} \in \mathcal{Q}_\alpha \mathcal{M}$ ,  $\alpha > 0$ . If  $\sum A_n(x)$  or  $\sum B_n(x)$  is strongly bounded at two points  $x_0$  and  $x_1$ ,  $|x_0 - x_1| \not\equiv 0 \pmod{\pi}$ , then  $\sum k^{1-\alpha} \rho_k^2 < \infty$ . These series are Fourier series of functions  $f, \tilde{f}$  which belong to  $L^2$  if  $\alpha \in (0, 1)$ . Moreover,  $f, \tilde{f} \in L^p$ ,  $2 < p < \frac{1}{\alpha}$ , if  $\alpha \in (0, \frac{1}{2})$ .*

#### 4. STRONG CONVERGENCE AND GENERALIZED VARIATION

Given a trigonometric series

$$(4.1) \quad \frac{a_0}{2} + \sum_{k=1}^n a_k \cos kx + b_k \sin kx$$

let  $s_n(x)$  and  $\sigma_n(x)$  denote the ordinary  $n$ -th partial sum and  $n$ -th Cesàro  $(C, 1)$  partial sum of (4.1), respectively. If (4.1) is a Fourier series of  $f \in L^1$ , we shall write  $s_n f$  and  $\sigma_n f$  for the partial sums  $s_n$  and  $\sigma_n$ .

We will consider the following classes of functions

$$\begin{aligned} \mathcal{S}^\lambda &= \{f \in L^1 : s_n f \rightarrow f \text{ } [I]_\lambda \text{ a.e.}\}, \\ \mathcal{S}^\lambda &= \{f \in C : s_n f \rightarrow f \text{ } [I]_\lambda \text{ uniformly}\}, \\ \mathcal{U} &= \{f \in C : s_n f \rightarrow f \text{ uniformly}\}, \end{aligned}$$

where  $C$  is the space of  $2\pi$ -periodic continuous functions.

For  $\lambda \geq 1$ , it is known (see [18]) that

$$\mathcal{S}^\lambda = \left\{ f \in L^1 : \sum_{k=1}^n k^\lambda \rho_k^\lambda = o(n) \right\}$$

and

$$\mathcal{S}^\lambda = \left\{ f \in C : \sum_{k=1}^n k^\lambda \rho_k^\lambda = o(n) \right\}.$$

By  $W$  we denote the class of *regulated functions*, i.e. functions possessing the one-sided limits at each point. Every regulated function is bounded and has at

most a countable set of discontinuities. Regulated functions have a particular role in the matter of everywhere convergence of Fourier series.

Important subclasses of the class  $W$  stem from various concepts of generalized bounded variation. In the sequel, let  $f(I) := f(b) - f(a)$  for arbitrary subinterval  $(a, b)$  of  $(0, 2\pi)$  and the supremum in defining sums below is always taken over all finite collections of nonoverlapping subintervals  $I_i$  of  $(0, 2\pi)$ .

According to N. Wiener [22], a function  $f$  is of  $p$ -bounded variation,  $p \geq 1$ , on  $[0, 2\pi]$  and belongs to the class  $V_p$  if

$$V_p(f) = \sup \left\{ \sum_i |f(I_i)|^p \right\}^{1/p} < \infty.$$

A function  $f$  is of  $\phi$ -bounded variation (L. C. Young [23]) on  $[0, 2\pi]$  and belongs to the class  $V_\phi$  if

$$V_\phi(f) = \sup \left\{ \sum_i \phi(|f(I_i)|) \right\} < \infty.$$

Here,  $\phi$  is a continuous function defined on  $[0, \infty)$  and strictly increasing from 0 to  $\infty$ .

Notice that by taking  $\phi(u) = u$  we get Jordan's class  $BV$ , while  $\phi(u) = u^p$  gives Wiener's class  $V_p$ .

A function  $f$  is of  $\Lambda$ -bounded variation (D. Waterman [20]) on  $[0, 2\pi]$  and belongs to the class  $\Lambda BV$  if

$$V_\Lambda(f) = \sup \left\{ \sum_i |f(I_i)| / \lambda_i \right\} < \infty,$$

where  $\Lambda = \{\lambda_n\}$  is a nondecreasing sequence of positive numbers tending to infinity, such that  $\sum 1/\lambda_n$  diverges.

In the case when  $\Lambda = \{n\}$ , the sequence of positive integers, the function  $f$  is said to be of *harmonic bounded variation* and the corresponding class is denoted by  $HBV$ .

$BV$  is the intersection of all  $\Lambda BV$  spaces and  $W$  is the union of all  $\Lambda BV$  spaces [9].

D. Waterman also introduced the notion of continuity in  $\Lambda$ -variation to provide a sufficient condition for  $(C, \alpha)$ -summability of Fourier series [21]. Let  $\Lambda^m = \{\lambda_{n+m}\}$ ,  $m = 0, 1, 2, \dots$ . A function  $f \in \Lambda BV$  is said to be *continuous in  $\Lambda$ -variation* (or to belong to  $\Lambda_c BV$ ) if  $V_{\Lambda^m}(f) \rightarrow 0$  as  $m \rightarrow \infty$ .

Clearly,  $\Lambda_c BV \subseteq \Lambda BV$ . Functions from  $\Lambda_c BV$  admit much better estimates of their Fourier coefficients (see [19, 12]).

The *modulus of variation* (Z. Chanturiya [7]) of a bounded function  $f$  is the function  $\nu_f$  whose domain is the set of positive integers, given by

$$\nu_f(n) = \sup \left\{ \sum_{k=1}^n |f(I_k)| \right\}.$$

The modulus of variation of any bounded function is nondecreasing and concave. Given a function  $\nu$  whose domain is the set of positive integers with such properties, then by  $V[\nu]$  one denotes the class of functions  $f$  for which  $\nu_f(n) = O(\nu(n))$  as  $n \rightarrow \infty$ . We note that  $V_\phi \subseteq V[n\phi^{-1}(1/n)]$  and  $W = \{f : \nu_f(n) = o(n)\}$  [7].

The relationship between Waterman's and Chanturiya's concepts was established in [1]. M. Avdispahić proved the following inclusions between Wiener's, Waterman's and Chanturiya's classes of functions of generalized bounded variation.

**Theorem 4.A** (cf. Theorem 4.4. in [2]).

$$\{n^\alpha\} BV \subset V_{\frac{1}{1-\alpha}} \subset V[n^\alpha] \subset \{n^\beta\} BV,$$

for  $0 < \alpha < \beta < 1$ .

The next two theorems are related to strong convergence and strong boundedness of Fourier series of regulated functions. As always, by  $\tilde{f}$  we denote the conjugate function of a function  $f$ .

**Theorem 4.1.** *Let  $\lambda \geq 1$ . Then*

- i)  $W \cap \mathcal{S}^\lambda = \mathcal{S}^\lambda$ .
- ii) *If  $f, \tilde{f} \in W$ , then  $f, \tilde{f} \in C$ .*
- iii) *If  $f \in \mathcal{S}^\lambda$  and  $\tilde{f} \in W$ , then  $\tilde{f} \in \mathcal{S}^\lambda$ .*
- iv) *If  $f \in HBV$  and  $\tilde{f} \in W$ , then  $f, \tilde{f} \in \mathcal{U}$ .*

*Proof.* i) Let  $f$  be an arbitrary function in  $W \cap \mathcal{S}^\lambda$ . Recall that  $\mathcal{S}^\lambda \subset \mathcal{S}$  [18, Theorem 1. (iii)]. Thus,  $\sum_{k=1}^n k \rho_k = o(n)$ , as  $n \rightarrow \infty$ . By [6, Theorem 3, p. 183 and Corrolary 2, p. 185],  $f$  can not have discontinuities of the first kind. It follows that  $f$  is a continuous function. Its Fourier series is  $(C, 1)$  uniformly summable. Therefore,  $f \in \mathcal{S}^\lambda$ . The converse,  $\mathcal{S}^\lambda \subseteq W \cap \mathcal{S}^\lambda$ , is trivial.

ii) Let  $f, \tilde{f} \in W$ . If there exists a point  $x_0$  such that, e.g.,  $f(x_0 + 0) - f(x_0 - 0) > 0$ , then by [24, Teorem 8.13, vol. I, p. 60]  $\tilde{S}_n(x_0, f) \rightarrow -\infty$ . Hence,  $\tilde{\sigma}_n(x_0, f) \rightarrow -\infty$ , which contradicts the fact that  $\tilde{\sigma}_n(x_0, f) = \sigma_n(x_0, \tilde{f}) \rightarrow \frac{1}{2} [\tilde{f}(x_0 + 0) + \tilde{f}(x_0 - 0)]$  [24, Fejér's theorem 3.4, vol. I, p. 89]. Therefore, function  $f$  is continuous. Analogously, the function  $\tilde{f}$  is continuous.

iii) Let  $\tilde{f} \in W$ . The conjugate series is  $(C, 1)$  summable to  $\tilde{f}$  a.e. [6, p. 524]. Therefore,  $f \in \mathcal{S}^\lambda$  implies  $\tilde{f} \in \mathcal{S}^\lambda$ . Hence,  $\tilde{f} \in W \cap \mathcal{S}^\lambda = \mathcal{S}^\lambda$  by i).

iv) By ii) above,  $f, \tilde{f} \in C$ . Now,  $f \in HBV \cap C$  implies uniform convergence of its Fourier series [20]. However,  $\tilde{f}$  being also continuous, its Fourier series is necessarily uniformly convergent as well, by [6, Theorem 1, p. 592].  $\square$

**Theorem 4.2.**

- i)  $\{n^{1/2}\} BV \cap C \subset \mathcal{S}^2$ .
- ii) *If  $f \in \{n^{1/2}\} BV$  and  $\tilde{f} \in W$ , then  $f, \tilde{f} \in \mathcal{S}^2$ .*
- iii) *If  $f \in V_2$ , then sequence  $\{s_n f\}$  is 2-strongly bounded.*

*Proof.* i) Let  $f \in \{n^{1/2}\} BV \cap C$ . Uniform convergence of the Fourier series follows from [20]. M. Avdispahić [2, Theorem 11.1] proved that the condition

$$(4.2) \quad \frac{1}{n} \sum_{k=1}^n k^2 \rho_k^2 = o(1) \text{ as } n \rightarrow \infty$$

is necessary and sufficient for continuity of  $f \in \{n^{1/2}\}_c BV$ . According to [11, Theorem 3.1] the equality  $\Lambda_c BV = \Lambda BV$  holds if and only if  $S_\lambda < 2$ , where  $S_\lambda$  is

the Shao-Sablin index defined by

$$S_\lambda := \limsup_{n \rightarrow \infty} \frac{\sum_{i=1}^{2n} \frac{1}{\lambda_i}}{\sum_{i=1}^n \frac{1}{\lambda_i}}$$

for every proper  $\Lambda$ -sequence  $\Lambda = \{\lambda_i\}$ . In case of  $\Lambda = \{i^{1/2}\}$ , we have

$$S_\lambda = \limsup_{n \rightarrow \infty} \frac{\sum_{i=1}^{2n} \frac{1}{\sqrt{i}}}{\sum_{i=1}^n \frac{1}{\sqrt{i}}} = \lim_{n \rightarrow \infty} \frac{\int_1^{2n} \frac{dx}{\sqrt{x}}}{\int_1^n \frac{dx}{\sqrt{x}}} = \lim_{n \rightarrow \infty} \frac{\sqrt{2n} - 1}{\sqrt{n} - 1} = \sqrt{2} < 2.$$

Therefore, (4.2) holds for  $f \in \{n^{1/2}\} BV \cap C$ . Since

$$\frac{1}{n} \sum_{k=1}^n k^2 |s_k f - s_{k-1} f|^2 = \frac{1}{n} \sum_{k=1}^n k^2 \rho_k^2 |\sin(kx + \theta_k)|^2 \leq \frac{1}{n} \sum_{k=1}^n k^2 \rho_k^2,$$

(4.2) and uniform convergence of  $\{s_n f\}$  imply that  $\{s_n f\}$  is 2-strongly convergent uniformly, i.e.  $f \in \mathcal{S}^2$ .

ii) If  $f \in \{n^{1/2}\} BV$  and  $\tilde{f} \in W$ , then  $f, \tilde{f} \in C$  by Theorem 4.1 ii). Now,  $f \in \mathcal{S}^2$  according to i) above. Moreover,  $\tilde{f} \in \mathcal{S}^2$  by Theorem 4.1 iii).

iii) If  $f \in V_2$ , then  $\frac{1}{n} \sum_{k=1}^n k^2 \rho_k^2 = O(1)$  [5, proof of Lema 3.1], and the sequence  $\{s_n f\}$  is 2-strongly bounded.  $\square$

*Remark 4.3.* In view of Theorem 4.A, the analogues of Theorem 4.2 i) and ii) are valid for Wiener classes  $V_p$ ,  $1 \leq p < 2$ , and Chanturiya classes  $V[n^\alpha]$ ,  $0 < \alpha < \frac{1}{2}$ .

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